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1995 J. Phys. A: Math. Gen. 28 7157

(http://iopscience.iop.org/0305-4470/28/24/015)

J. Phys. A: Math. Gen. 28 (1995) 7157–7180. Printed in the UK

Complete Poincaré sections and tangent sets

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Received 26 June 1995, in final form 25 September 1995

Abstract. Trying to extend a local definition of a surface of a section, and the corresponding Poincaré map to a global one, one can encounter severe difficulties. We show that global transverse sections often do not exist for Hamiltonian systems with two degrees of freedom. As a consequence we present a method to generate the so-called W-section, which by construction will be intersected by (almost) all orbits. Depending on the type of tangent set in the surface of the section, we distinguish five types of W-sections. The method is illustrated by a number of examples, most notably the quartic potential and the double pendulum. W-sections can also be applied to higher dimensional Hamiltonian systems and to dissipative systems.

1. Introduction

Poincaré [1] introduced the idea of studying a map instead of the flow in order to show the existence of periodic orbits in the restricted three-body problem. His work was completed and extended by Birkhoff [2], who initiated the study of smooth area preserving maps in [3]. Hénon and Heiles were among the first to obtain pictures of the Poincaré surface of a section for non-integrable systems by numerically integrating the flow [4]. Since then the method has become a standard tool for the analysis of nonlinear dynamical systems, and it is introduced in most textbooks (see e.g. Guckenheimer and Holmes [5], Lichtenberg and Liebermann [6] or Ozorio de Almeida [7]).

It is commonly known that a surface of a section (section in the following) should be transverse to the flow. Transversality of a section is indeed a very nice property, because the induced Poincaré map will be well defined and smooth. We will show that in many cases, however, it is impossible to obtain a transverse section for the whole flow. This fact has mainly been ignored in the literature, despite the fact that almost all examples of global sections for a time-independent Hamiltonian system with two degrees of freedom are *not* transverse.

The question of how to construct a section in such a way that the induced Poincaré map is well defined and smooth, although the section is not transverse, has already been resolved by Birkhoff [2]. His essential points are that (i) the non-transverse set has to be invariant under the flow, and (ii) the section has to be complete, i.e. (loosely speaking) repeatedly hit by every orbit.

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0305-4470/95/247157+24$19.50

Our construction of a section, which we will call a W-section, because it is generated by some function W on phase space, will give a weak form of the second property automatically. Sometimes the first property also holds for a W-section, so that most of the work of proving the induced Poincaré map to be smooth is already done. In this way a W-section can give a Poincaré map in the sense of Birkhoff—sometimes it even yields a transverse section. A W-section can be used in n dimensions. Nevertheless we will formulate our results for two degrees of freedom, and finally comment on the higher dimensional case.

Dealing with systems that, due to symmetries, have explicitly solvable periodic orbits, there is a good chance of obtaining a Birkhoff section. If this is not the case, but for practical purposes we nevertheless want to obtain an explicitly given section condition, we have to sacrifice Birkhoff's first condition, and therefore the smoothness of the induced Poincarè map. Thus it is necessary to discuss the possible types of discontinuities in a non-transverse section in some detail.

We encourage the use of these non-smooth maps, but insist that the corresponding sections should be at least asymptotically complete (to be defined below). Under this assumption they can be quite useful in order to obtain a global overview of the dynamics (although they might be less useful as a mathematical object, because no fixed point theorems etc apply). Using a W-section guarantees the (asymptotic) completeness of the section, and therefore the induced Poincarè map will be (almost) well defined. Depending on the properties of the non-transverse set we distinguish five types of W-sections including the ideal (everywhere transverse) and Birkhoff sections.

Generally we distinguish between the section with its properties and the induced Poincaré map, although this distinction is not always very clear cut. We try to deduce properties of the map from the properties of the section. If a section is not complete we consider it dangerous to use it. Investigating the return time we suggest a method to numerically detect whether a given section is of this unpleasant type. This will be illustrated in an example. For all the five better types of sections we also give examples, where we always use a W-section.

A transverse section should always be a smooth and closed submanifold. For cases where there does not exist a transverse section, Birkhoff introduced a surface of a section as a submanifold with boundaries, which are invariant under the flow. Because we are going to discuss sections with a non-transverse set that is *not* invariant under the flow we prefer to consider the section to always be a closed submanifold. Moreover, we feel that the geometric situation in phase space with respect to the energy surface becomes more clear by this. When we consider Birkhoff sections we stay with this habit, although it is redundant in this case as it gives a duplication of manifolds with boundaries glued together.

The paper is organized as follows. Following a general introduction to Poincaré sections in section 2, we state conditions such that the section can be transverse and complete. Then we show that in important cases it can be impossible to find complete and transverse sections. In section 4 we discuss the implications of a non-transverse section for the discontinuities of the Poincaré map. We review the construction by Birkhoff, which assures a smooth Poincaré map in spite of non-transversality. In our general construction of W-sections presented in section 5, we obtain completeness almost for free, but might lose the transversality of the section and the smoothness of the Poincaré map. Depending on the properties of the non-transverse set in the section, we distinguish five types of W-sections. For each of them we discuss an example in section 6. We finally comment on non-Hamiltonian and higher dimensional cases.

2. Poincaré sections

Consider a Hamiltonian system with two degrees of freedom and configuration space Q. We will restrict our attention to time-independent Hamiltonians $H(x) = h$ defined on phase space T^*Q. When necessary we will introduce (local) canonical coordinates $x = (q, p) = (q_1, q_2, p_1, p_2)$. Fixing h to a non-critical value we obtain a vector field ξ without singularity on the three-dimensional energy surface

$$\mathcal{E}_h = \{x \in T^*Q | H(x) = h\} \tag{1}$$

which we assume to be a compact manifold. If it has more than one component each of them can be discussed separately. The vector field ξ in a local coordinate system is given by Hamilton's equations

$$\dot{x} = (\dot{q}, \dot{p}) = (\partial H/\partial p, -\partial H/\partial q). \tag{2}$$

The corresponding flow is denoted by Φ. To reduce the continuous dynamics to a Poincaré map we introduce a two-dimensional section Σ_h, with the corresponding Poincaré map P. It can always be defined in the neighbourhood of a periodic orbit which intersects Σ_h transversely (see e.g. [8, 9, 5]) by

$$\begin{aligned} P : \Sigma_h &\to \Sigma_h \\ x &\mapsto P(x) = \Phi^\tau(x) \end{aligned} \tag{3}$$

with the first return time τ

$$\tau(x) = \min\{t \in \mathbb{R} | t \rangle 0, \Phi^t(x) \in \Sigma_h\}. \tag{4}$$

In the ideal case we can do this not only locally but for the whole flow such that the following properties hold for the section:

manifold: Σ_h is a closed submanifold of $\mathcal{E}_h$ of codimension one, i.e. a Riemann surface;

transversality: ξ is everywhere transverse to Σ_h;

Σ-completeness: every orbit starting from Σ_h has at least one more point in Σ_h for finite positive and negative times;

$\mathcal{E}$-completeness: every orbit in $\mathcal{E}_h$ does intersect Σ_h.

We will call a section complete if it is Σ-complete and $\mathcal{E}$-complete. Notice that all four items are properties of the section and *not* of the Poincaré map P. The two definitions encompass forward and backward time evolution. Our arguments will usually be presented only for the positive time direction, but they can easily be translated to the time reversed case.

We will see that in important cases it is impossible to obtain all four properties. For the purpose of obtaining a global overview of the dynamical system, the last two properties are the most important. We will suggest a recipe to obtain sections that are complete, but which may lack the first two properties.

In the following we consider only sections Σ_h which can be defined by an equation $S(x) = 0$ (the section condition) on phase space, where S is a smooth function. Denote the hypersurface in phase space defined via S by

$$\mathcal{S} = \{x \in T^*Q | S(x) = 0\}. \tag{5}$$

This definition is most convenient for analytical and numerical purposes, for example, Hénon's method to obtain the exact intersection of a trajectory and the section [10] can readily be generalized to a section condition as given above by reparametrizing the flow with S. The types of section conditions commonly used in the literature (e.g. $q_1 = 0$) are obviously special cases of the above more general form. As we will see later, examples where $\mathcal{S}$ is not a manifold but instead a collection of manifolds, which intersect, arise quite naturally for W-sections, when the section condition can be split into factors and each of them can become zero. In such cases we have to discuss each manifold, defined by a section condition given by one of the factors separately.

The restriction of the hypersurface $\mathcal{S}$ to the energy (hyper)surface $\mathcal{E}_h$ gives the two-dimensional section

$$\Sigma_h = \{x \in T^*Q | H(x) = h, S(x) = 0\} = \mathcal{S} \cap \mathcal{E}_h. \tag{6}$$

If $\mathcal{S}$ is a manifold, then also this surface usually is a manifold, because generically $H(x)$ and $S(x)$ are independent. If the manifold property is strictly required it should of course be checked by verifying rank $(\nabla S \nabla H) = 2$ on Σ_h.

Our strategy will be to deduce properties of the Poincaré map P from the properties of the section. Most notably, we will determine under which conditions P is smooth and/or well defined. If P is not smooth we, moreover, classify the possible types of discontinuities.

Let us briefly consider the question of how to represent Σ_h in $\mathbb{R}^2$. By definition Σ_h is embedded in the four-dimensional phase space. Introducing local coordinate systems, we must in principle cover Σ_h by a collection of charts. In practice one usually tries a projection of Σ_h onto a two-dimensional plane. A natural choice of this plane is one canonical pair of the original variables, which yields an area preserving representation of P in $\mathbb{R}^2$ if the section condition is $q_i =$ constant or $p_i =$ constant [11]. The problem is that the projection of a closed surface onto a plane is at least twofold.

Following Birkhoff, one usually considers that part of Σ_h with $\dot{S} \geqslant 0$ (or $\leqslant 0$) as the section. In many examples shown in the literature this approach works and a projection onto a canonical plane can be found. Obviously, this method fails if the projection is more than twofold. Also, in the following there will be cases where orbits intersect Σ_h only once, either with $\dot{S} > 0$ or $\dot{S} < 0$. Thus we prefer to consider Σ_h as a closed surface instead of a surface patch bounded by the set $\dot{S} = 0$. The most obvious way to solve this problem is not to create it in the first place: instead of projecting Σ_h into $\mathbb{R}^2$, it can be mapped into $\mathbb{R}^3$. This approach will be studied in a forthcoming paper. However, in the examples we do use projections to $\mathbb{R}^2$, in order to present pictures.

3. Ideal sections ...

Birkhoff found that a necessary and sufficient condition for an ideal section is the existence of a global angle coordinate in the energy surface such that the vector field then gives $\dot{\phi} > 0$ (or < 0) always [12]. The section $\phi =$ constant then has all the desired properties, including the smoothness of the induced map P. The converse formulation (also for arbitrary non-Hamiltonian systems) is that $\mathcal{E}_h$ is the result of a suspension of some given diffeomorphism of Σ_h onto itself (see e.g. [13, 9]).

Already the requirement that a global angle variable has to exist poses a strict requirement on the topology of the energy surface. The topology of $\mathcal{E}_h$ must allow for the embedding of a two-dimensional Riemann surface Σ_h such that it does *not* divide $\mathcal{E}_h$ into two parts. If the vector field ξ is transverse to Σ_h the flow always intersects Σ_h in the same direction. If Σ_h divides $\mathcal{E}_h$ into two parts the section is obviously not transverse: using the direction of the flow there is a notion of the inside and outside of Σ_h so, if ξ is everywhere transverse to Σ_h, orbits that enter the inside could never leave it, which is impossible for Hamiltonian flows.

An ideal section is obviously possible if $\mathcal{E}_h$ has the structure of a direct product with S^1, the particular cases $\mathcal{E}_h \simeq S^1 \times S^2$ and $\mathcal{E}_h \simeq S^1 \times T^2 \simeq T^3$ do occur in mechanical problems (see Smale [14] for a discussion of the topology of Hamiltonian systems)†. If $\mathcal{E}_h$ is a direct product $S^1 \times \Sigma_h$, and we are able to introduce an increasing angle, then after an appropriate rescaling of time the system has been reduced to one and a half degrees of freedom, where the Hamiltonian is driven periodically in time. In the neighbourhood of a periodic orbit this procedure is always possible (see e.g. [7]), but we want to require it globally, and many flows will not have this property.

Most examples of ideal section known to us are of the stroboscopic type (see e.g. [5, 6]), i.e. the vector field depends periodically on time t with period T, and the section condition is given by $(t \bmod T) = 0$. The Poincaré map is then especially simple, because the return time $\tau(x)$ is independent of x.

3.1. ... can be impossible

Continuing the above intuitive argument easily yields the result that there cannot be an ideal section for S^3 (and of course also not for $\mathbb{R}^3$, thinking about non-Hamiltonian flows). It is known that *every* embedded (orientable) Riemann surface Σ_h does divide S^3 ($\mathbb{R}^3$) into two parts. Since the energy surface of many Hamiltonian systems for energies slightly above the critical energy of a stable equilibrium is S^3, this case is of utmost importance.

For reversible systems the following argument shows that time reversal symmetric surfaces of a section which consist of one component are always non-transverse.

If the Hamiltonian is quadratic in the momenta the system possesses time reversal symmetry $(p \to -p)$, $(t \to -t)$. If S contains only even powers of the momenta it is also invariant under time reversal. A common section condition of this type is $S(q) = 0$, i.e. S is a function on configuration space only. Let $(q_0, p_0) \in \Sigma_h$ be an initial condition for the solution $(q(t), p(t))$. Because of the symmetry $(q(-t), -p(-t))$ is also a solution of the system. Evaluating $\dot{S}$ and taking the invariance of S and H into account gives

$$\dot{S}|_{(q_0,p_0)} = -\dot{S}|_{(q_0,-p_0)} \tag{7}$$

which shows that the direction of intersection of the flow with Σ_h has both signs. Since we assume Σ_h to be a manifold without boundary and one component only, somewhere we must have $\dot{S} = 0$, and we conclude that the section cannot be transverse.

Note that defining a section for an integrable system by using action angle variables does not allow the introduction of increasing angle variables in the desired form, because obviously the topology of the energy surface as well as the vector field defined on $\mathcal{E}_h$ cannot be changed. Moreover, action angle variables can usually not be introduced globally. They only cover parts of phase space where there is a compatible foliation by Liouville tori and they become singular on stable periodic orbits.

Using more technical arguments from homotopy theory of fibre bundles one can show the non-existence of ideal sections for all Hamiltonian systems with two degrees of freedom and quadratic kinetic energy except for cases where $\mathcal{E}_h \simeq S^\times S^2$ and $\mathcal{E}_h \simeq T^3$. This will be presented in a forthcoming paper. Even if the energy surface has that topology, Birkhoff's second condition ($\dot{\phi} > 0$) is a strong restriction on the flow. Ideal sections are therefore exceptional.

† Throughout this paper S^n denotes the n-dimensional sphere, T^n the n-dimensional torus, D^1 the interval, D^2 the disk, and D^3 the three-dimensional ball.

4. Non-transverse sections

Let us study the tangent set of Σ_h where the Hamiltonian vector field ξ is not transverse to Σ_h. Since we assume Σ_h to be a smooth manifold, ∇H and ∇S are linearly independent; thus a tangent vector to Σ_h is orthogonal to both vectors. The Hamiltonian flow is by construction orthogonal to ∇H so that we can define the *tangent set* $\Omega \subset \Sigma_h$ by

$$\Omega = \{x \in \Sigma_h | \dot{S} = \nabla S \cdot \xi = 0\}. \tag{8}$$

In general Ω is of codimension two with respect to $\mathcal{E}_h$ and therefore a one-dimensional subset of Σ_h. If we cross Ω transversely on Σ_h the direction of intersection, i.e. $\dot{S}$, changes its sign.

If Ω is non-empty, the Poincaré map P from Σ_h onto itself can be discontinuous. Consider a part ω of Ω for which $\ddot{S} = \nabla\dot{S} \cdot \xi \neq 0$, i.e. which is not invariant under the flow. This implies that ξ is transverse to ω (it nevertheless is *not* transverse to Σ_h). Let us denote a small one-sided neighbourhood of ω in Σ_h by A such that $\xi|_\omega$ points away from A (see figure 1). P then maps A to the other side of ω. Let us denote the image $P(A)$ by B. The return time $\tau(x)$ for $x \in A$ approaches zero if x approaches ω. However, initial points $x \in B$ close to ω are carried away from Σ_h by the flow and they have $\tau(x) > 0$. Therefore P is discontinuous along ω (see the centre of figure 1). The line ω itself is mapped together with B in accordance with definition (4).

Another kind of discontinuity occurs at the pre-image of ω. Considering the pre-images of A and B the situation is analogous to the above: the pre-image of B is A and the pre-image of A, in general, is some patch of Σ_h which does not contain a tangent set. Using the inverse map, ω is mapped with A. Let us now investigate the images of the neighbourhood of the pre-image $P^{-1}(\omega)$, which consists of $P^{-1}(A)$ and another patch denoted by C, see figure 1. Because of the continuity of the flow, $P^{-1}(A)$ and C do evolve 'side by side' in phase space; but C will miss the section at the moment that A first hits it. Therefore P is discontinuous along $P^{-1}(\omega)$. However, because of the continuity of the flow the images of A and C must join again. Even though the map is discontinuous, most notably KAM tori will continue smoothly across ω and its pre-image. In particular, this implies that on the computer screen one does not notice this kind of discontinuity of the map. This will be demonstrated in one of the examples (see figure 6 later).

From the numerical point of view the discontinuity does not pose a severe problem, since the calculation of P is performed pointwise. Only the determination of intersections of orbit segments that are close to the tangent set is susceptible to round off errors. Also the reparametrization of the flow with S, in order to find the exact intersection, then becomes difficult. The calculation of the linearized map by numerically differentiating the map, i.e. by mapping close by trajectories, does become a problem, but this is not the method of choice anyhow. If instead the variational equation is integrated along the given orbit, the problems in determining the stability vanish. In applying Newtons method to find periodic orbits using the Poincaré map, care has to be taken in order to avoid a crossing of the pre-image of Ω. This can be achieved by monitoring $\tau(x)$.

From the analytical point of view we would like to construct smooth maps. We have seen that discontinuities occur if ξ is transverse to Ω. If on the contrary ξ is tangential to Ω, it is possible to obtain a smooth map.

Let us define $\Lambda \subset \Omega$ as being the subset of Ω which is invariant under the flow Φ, i.e.

$$\Lambda = \{x \in \Omega | \Phi^t(x) \in \Omega \ \forall t\}. \tag{9}$$

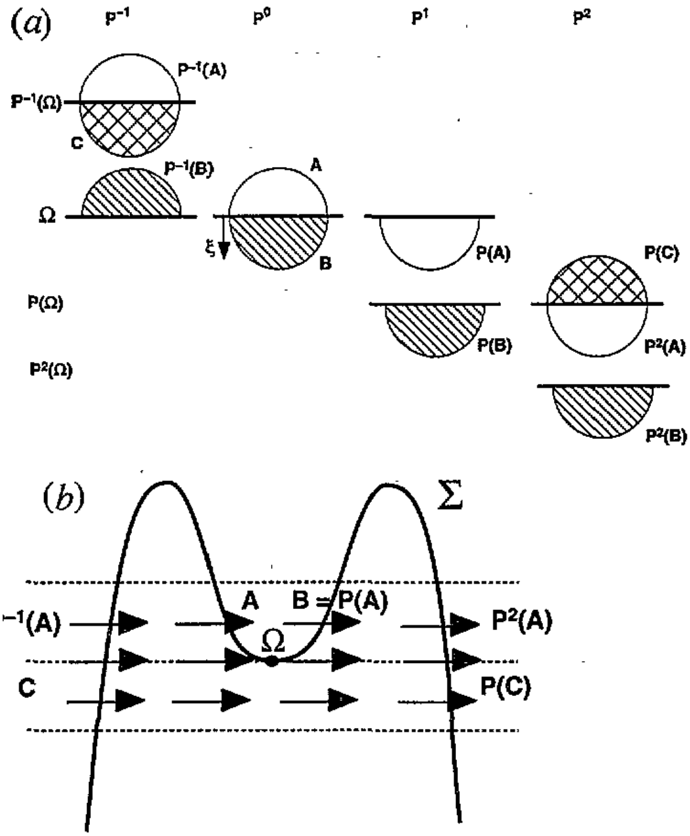


Figure 1. The properties of the Poincaré map in the neighbourhood of a part of the tangent set Ω which is not invariant under the flow. (a) The initial sets A and $B = P(A)$ close to Ω are indicated in the second column. Each row represents the same part of the section, while each column shows a different iteration step of the sets A and B. The projection of the vector field ξ onto the section points into B on Ω. In the third column the images $P(A)$ and $P(B)$ are shown. If one restricts attention to the part of Σ_h with the same sign of $\dot{S}$ as in B, this discontinuity of P disappears because $A = P^{-1}(B)$ has the opposite sign, but there always is a discontinuity along the pre-image of Ω, indicated in the first column (a 'real' picture of this singularity is shown in figure 6). (b) The same situation as in (a) viewed 'from the side'. Instead of looking onto the tangent set Ω in Σ_h a cross section of Σ_h together with the flow, indicating the tangency at Ω, is shown.

For practical purposes we can rephrase the definition:

$$\Lambda = \{x \in \mathcal{E}_h | \frac{d^n S}{dt^n} = 0, n \in \mathbb{N}\}. \tag{10}$$

Obviously, $n = 0$ implies $\Lambda \subset \Sigma_h$ and for $n \leqslant 1$ we find $\Lambda \subset \Omega$. Generically, for $n \leqslant 2$ Λ should be of codimension three with respect to the energy surface, i.e. points, while also requiring the third derivate to vanish we would get an empty set. Therefore the vanishing of the second derivate on a line in Ω is already a strong indication of the existence of an invariant tangent set. Nevertheless, one should verify that trajectories starting in Λ stay in the surface of section. As we will see in the next section, Λ will be composed of periodic orbits of Φ. Let us consider the neighbourhood in $\mathcal{E}_h$ of a periodic orbit in Λ, in order to determine its impact on completeness.

Stable periodic orbits in Λ are almost always nice in the sense that they are not an obstacle to completeness. Neighbouring orbits wind around the periodic orbit. Only in rare cases is the winding number of the periodic orbit zero when measured with respect to the section. We exclude this case in the following. In all other cases every neighbouring orbit will intersect the section; we will call these periodic orbits 'winding'.

Unstable periodic orbits in Λ can cause problems if their local invariant manifolds are not winding with respect to the section. In this case orbits can approach the section along the stable manifold and they will not produce an intersection in finite time: the section is not Σ-complete or not $\mathcal{E}$-complete or neither. These periodic orbits will be called 'non-winding'. If the unstable periodic orbit has non-orientable local invariant manifolds, i.e. it is hyperbolic with reflection, the orbit is winding because the section is orientable.

Note that if we have a non-winding unstable orbit in Λ there can be another type of discontinuity in P. If the stable invariant manifold does intersect Σ_h in some place (by assumption it does not intersect close to the periodic orbit) P will be discontinuous at this line, and the return time τ approaches ∞ close to it. If the stable manifold does not intersect Σ_h at all, Σ_h is not $\mathcal{E}$-complete. Conversely, if there are only winding stable or unstable orbits in Λ and $\Lambda = \Omega$, P can be smooth.

Let us introduce two weaker notions replacing completeness. A section is called:

Σ^∞-complete if every orbit starting in Σ_h which does not return in finite time, has a $t \to \pm\infty$ limit set in Σ_h;

$\mathcal{E}^\infty$-complete if every orbit in $\mathcal{E}_h$ which does not reach Σ_h in finite time, has a $t \to \pm\infty$ limit set in Σ_h.

We will call a section asymptotically complete, if it has both properties.

With these definitions we can summarize the facts about the different types of singularities in the Poincaré map.

(i) If $\omega = (\Omega \backslash \Lambda) \neq \emptyset$ there is a discontinuity along ω and $P^{-1}(\omega)$. Restricting the section to the part of Σ_h with one sign of $\dot{S}$ removes the singularity at ω. The singularity at $P^{-1}(\omega)$ can of course not be removed and it is characterized by a finite jump in τ (see figure 1).

(ii) If there are non-winding unstable orbits in Λ the section is at most Σ^∞-complete. There is a discontinuity in P where $\tau \to \infty$. We consider P as 'almost well defined' if the section is Σ^∞-complete and $\mathcal{E}^\infty$-complete.

(iii) There can be an especially bad third kind of singularity if Σ_h is not even $\mathcal{E}^\infty$-complete because there might exist unstable periodic orbits that have no points at all in Σ_h with stable manifolds which do intersect with Σ_h. Again we find $\tau \to \infty$.

The construction of W-sections in the next section will avoid this last type of singularity, which might lead to a fractal set of singularities in P (see figure 4 later for an example).

Birkhoff [2] showed that a necessary condition to obtain a smooth Poincaré map in the non-transverse case is $\Omega = \Lambda$, i.e. the non-transversality is concentrated in the invariant set. He defined a section to be (i) complete, (ii) (if not ideal) with a finite number of boundaries which are invariant under the flow, and (iii) bounded in such a way that the angle of intersection of ξ and Σ_h goes linearly to zero when approaching Λ on Σ_h. The third condition is needed in order to define a return time $\tau(x)$ for $x \in \Lambda$, because the simple definition (4) does not work, since the orbit never leaves Σ_h. As a result of this definition Birkhoff showed that the induced Poincaré map is smooth and can be smoothly extended onto the invariant tangent set. Moreover, he proved that, for certain systems, sections with all of the above properties do exist in principle. As Siegel and Moser remark in [11], it is the existence of invariant boundaries which makes a fixed-point theorem possible in the non-transverse case.

If the Hamiltonian system permits the explicit solution for a periodic orbit, usually due to symmetry, one might be able to construct a section according to Birkhoff's definition, such that the section condition can be given in simple explicit form. The main difficulty is to prove completeness. The method of W-sections introduced in the next section is a tool with which one can do this.

In general, the problem with Birkhoff's construction is that one needs to know about the periodic orbits in order to choose the section condition, while in practice one wants to construct a section in order to learn about the periodic orbits. Since our main goal is to construct complete sections, we will allow for the non-invariance of the tangent set.

5. Complete sections

As a general method for constructing a section for bounded Hamiltonian systems we suggest one makes use of the idea of Lorenz [15], to take the extrema of a bounded quantity as a section condition. We are not aware of an extension of his idea to the study of Hamiltonian systems. Consider an smooth function W on phase space which maps $\mathcal{E}_h$ to an interval D^1

$$W(\mathcal{E}_h) = D^1 \subset \mathbb{R} \qquad \dot{W}(\mathcal{E}_h) \not\equiv 0. \tag{11}$$

Let us choose as a section condition

$$S_W(x) = \dot{W}(x) = 0 \tag{12}$$

and call the corresponding section Σ_w. W can be thought of as a continuous signal of the system that 'generates' the section.

Under the assumption that $W(x)$ and the vector field ξ are analytic on $\mathcal{E}_h$ we obtain the following theorem which is independent of the number of degrees of freedom.

Theorem 1. A W-section is an at least asymptotically complete section.

Proof. Consider a trajectory $\Phi^t(x)$. From the assumptions it follows that $W(\Phi^t(x))$ is also analytic and bounded. Looking at the behaviour of the signal $W(\Phi^t(x))$ as time goes to $\pm\infty$ the assumptions only allow for the following behaviour concerning extremal points (compare with figure 2(a)):

(A) $W(\Phi^t(x))$ repeatedly passes through extremal points;

(B) $W(\Phi^t(x))$ is constant, which implies that $W(\Phi^t(x)) = \text{constant}$ for all times t, and we have $\dot{W}(\Phi^t(x)) \equiv 0$;

(C) $W(\Phi^t(x))$ is asymptotically approaching some constant value in a monotonic way.

We now interpret the behaviour of the signal in terms of the dynamical system and the section Σ_w. Let us denote the sets of trajectories which generate the above signals by $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

(A) Set $\mathcal{A}$ does not pose a problem in terms of completeness of a W-section, because the signals repeatedly pass through extrema and the trajectories therefore intersects Σ_w an infinite number of times. This is the generic behaviour of trajectories because we required $\dot{W}(\mathcal{E}_h) \not\equiv 0$.

(B) Set $\mathcal{B}$ by definition represents all orbits which are in Σ_w, i.e. $\mathcal{B} = \Lambda$.

(C) Set $\mathcal{C}$ consists of trajectories which are asymptotic to some iso-surface $\mathcal{W}_c = \{x | W(x) = c\}$ or some subset of this surface. Since $\mathcal{E}_h$ is compact the limit set Γ of $\Phi^t(x)$ exists and since $W(\Phi^t(x)) \to c$ we have $\Gamma \subseteq \mathcal{W}_c$. Of course by definition we also have $\Gamma \subseteq \Lambda$.

Since $\mathcal{E}_h = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ a W-section is complete if $\mathcal{C} = \emptyset$ and asymptotically complete otherwise. □

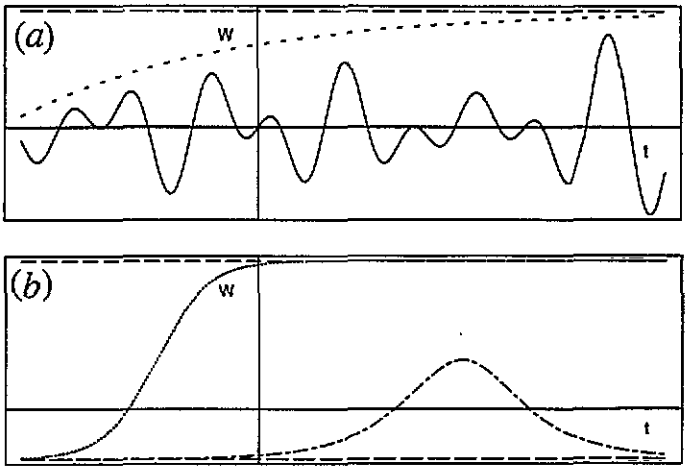


Figure 2. (a) Types of signals for a function W: the full curve shows the generic case of a signal, which repeatedly has extrema; the dashed curve indicates the signal of an orbit which lies entirely in the section; and the dotted curve shows the signal of an orbit asymptotic to the previous one in forward time. (b) The schematic signal of homo and heteroclinic motion: the two dashed curves represent unstable periodic orbits which lie entirely in the section; heteroclinic motion can be identified as the dotted curve while the remaining curve stands for homoclinic motion. Notice that the sketch represents worst case, non-winding periodic orbits. The approach to a winding orbit would give a signal which oscillates an infinite number of times around its final value.

Corollary 1. If $\Lambda = \emptyset$ the W-section is complete.

Proof. Because $\mathcal{C}$ is asymptotic to $\mathcal{B}$, $\mathcal{C}$ must be empty if $\mathcal{B}$ is empty. □

Corollary 2. If Λ contains only invariant sets which are not limit sets (except for the members of the set themselves), then the W-section is complete.

Proof. The limit set of $\mathcal{C}$ is in $\mathcal{B}$, but $\mathcal{B}$ does not contain limit sets, so $\mathcal{C}$ is empty. □

Corollary 3. The W-sections for 2 degree of freedom systems can be classified by the tangent set Ω and its invariant component Λ as follows:

(i) $\Omega = \emptyset$, the W-section is ideal;

(ii) $\Omega \neq \emptyset$ and $\Omega = \Lambda$ and Λ does not contain non-winding periodic orbits, the W-section is a Birkhoff section if the linearity condition holds;

(iii) $\Omega \neq \Lambda$ and Λ does not contain non-winding orbits, the W-section is complete;

(iv) $\Omega \neq \Lambda$ and Λ contains one non-winding unstable periodic orbit, the section is $\mathcal{E}$-complete and Σ^∞-complete;

(v) $\Omega \neq \Lambda$ and Λ contains more than one non-winding unstable periodic orbit, the section is Σ^∞-complete and at least $\mathcal{E}^\infty$-complete.

Proof. Case (i) is obvious, because $\Omega = \emptyset$ gives the transversality and from corollary 1 we obtain the completeness. To obtain the other cases we show that Λ can only contain periodic orbits. The only invariant sets embeddable in any surface $\mathcal{W}_c$ are two-tori or periodic orbits. The case when two-tori are found in Λ is exceptional. In an integrable system this would either imply that we had chosen W to be a constant of motion (which we explicitly excluded in our definition (**11**)), or we would have the same situation as in the non-integrable case: Σ_w would necessarily be a set of surfaces, one of them being a KAM torus (otherwise all of the energy surface would have to asymptotically approach that torus, which is obviously impossible). Accidently, describing a KAM torus by an equation $\dot{W} = 0$ is highly improbable, and, in any case, we would not lose trajectories. We now restrict attention to the components of Σ_w that are not KAM tori.

Therefore the only invariant sets in Λ can be periodic orbits. We already noted in the last section that the section is complete if there are no non-winding orbits in Λ, and from this we obtain (ii) and (iii). If there are non-winding unstable orbits in Λ the section cannot be Σ-complete. The first intersection of the stable manifold of such an orbit with Σ_w is a *line. Orbits starting on this line do not return in finite time. If the two lines of the first* intersection of the stable and unstable manifold of the same orbit intersect in Σ_w, there is a special homoclinic orbit as shown in figure 2(*b*), which intersects the section only once. Because this is the worst case, the section (iv) is still $\mathcal{E}$-complete.

Only certain heteroclinic orbits are an obstacle for the W-section to be $\mathcal{E}$-complete. Their $t \to \pm\infty$ limit sets are in Λ, and the generated signals are monotonic, see figure 2(*b*). Obviously there must be at least two non-winding unstable orbits in Λ for this to be possible, which yields case (v). □

That, indeed, all these cases can occur will be shown in the examples. In addition there will be examples of what can go wrong if we do not choose a section which ensures (asymptotic) completeness like a W-section does.

If $\Lambda = \emptyset$ *we automatically have a complete section. An important observation is* that this property can be checked purely locally, i.e. we do not need to know anything about the solutions of our differential equation. Usually this is achieved by verifying that $\ddot{S}(x) \neq 0$ on Ω. By a suitable choice of W we can achieve $\Lambda = \emptyset$ and thus avoid possible problems with non-winding periodic orbits. This can usually be accomplished by choosing a function W such that $\dot{W}$ does not respect symmetries of the system. The determination of the properties of periodic orbits in Λ is much harder, because it requires the solution of differential equations.

Before passing to the examples, notice that a W-section is set up independently of the topology of the energy surface and of the accessible configuration space Q_h, which, in general, both change with a change in the energy or some other parameters. We, therefore, do not have to reconsider the definition of the section. Also we find that a change in *the topology of the energy surface is reflected in a change of the topology of the section,* because by definition an equilibrium point x^* will be in Σ_w (we have $\dot{W}(x^*) = 0!$) and also this surface has a singularity (rank $(\nabla S, \nabla H)|_{x^*} < 2$).

6. Standard W-sections and examples

For a standard Hamiltonian $H = T + V$ with a positive definite kinetic energy T of quadratic form and potential $V(q)$, there are a number of standard choices for W:

zero-force	$W = p_i, S = \dot{W} = \dot{p}_i = 0$
zero-velocity	$W = q_i, S = \dot{W} = \dot{q}_i = 0$ (only if q_i is not an angle!)
zero-acceleration	$W = v_i, S = \dot{W} = \dot{v}_i = \ddot{q}_i = 0$
equipotential	$W = V, S = \dot{W} = \dot{V} = -\dot{T} = 0$

The first two just amount to taking the zeroes of one component of the right-hand side of Hamilton's equations as a section condition; the third one arises naturally if we consider the second-order Euler–Lagrange equations. In the last case the projection of the trajectory into the configuration space is a tangent to an equipotential line at the moment of intersection. The zero-acceleration section can only give new results if the kinetic energy is non-trivial. Note that taking the zeroes of the right-hand side of a differential equation as a section condition is reminiscent of the method of zero-isoclines, which graphically locates fixed points in two-dimensional flows.

If T does not depend on q, the zero-force section gives a section condition purely in configuration space: $S(q) = \dot{p}_i = -\partial V/\partial q_i \equiv -V_{q_i} = 0$. In this case a projection of parts of Σ_w with $\dot{S} > 0$ or < 0 to the symplectic coordinate plane (q_j, p_j), $i \neq j$ can give a representation in $\mathbb{R}^2$. However, in many cases this projection will not be invertible. Choosing coordinates on Σ_w is a non-trivial problem for many zero-force sections. In the ideal case one can introduce V_{q_i} as a new coordinate Q_i, such that the projection to (Q_j, P_j) gives an area preserving representation in $\mathbb{R}^2$ (see the examples).

Zero-velocity and equipotential sections are linear in the momenta for standard Hamiltonians. Therefore p can be determined if the potential energy is given, and there are at most two solutions. The projection of Σ_w onto configuration space is, therefore, a double covering. This projection of Σ_w neither gives an area preserving representation of the Poincaré map in $\mathbb{R}^2$, nor is its boundary (if it exists) the tangent set. It has the advantage, however, that it is easily obtained, and often, therefore, might be the first choice.

6.1. Ideal sections

One example for an ideal section is provided by the motion of a charged particle in a doubly-periodic magnetic field $B(x, y)$ of constant sign, which was suggested by Birkhoff [2]. Here we use a W-section to obtain the result, and extend it by the addition of a non-constant potential. Due to the periodicity of B, configuration space can be considered to be a two-torus T^2 and therefore $\mathcal{E}_h \simeq T^3$. Introducing the vector potential $A(x, y)$ the Lagrangian reads

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - V_0 + A(x, y) \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \tag{13}$$

with a constant potential V_0. In the present problem it is more convenient to use the Euler–Lagrange equations

$$\ddot{x} = -B(x, y)\dot{y} \qquad \ddot{y} = B(x, y)\dot{x} \tag{14}$$

instead of Hamilton's equations. Let us, therefore, consider the zero-acceleration section $S = \ddot{x} = -B(x, y)\dot{y} = 0$. Since B has constant sign by assumption, the section condition is only fulfilled if $\dot{y} = 0$. The tangent set is given by $\ddot{y} = B(x, y)\dot{x} = 0$ on Σ_w, but because of energy conservation the kinetic energy $2(h - V_0)$ is always positive, i.e. $\dot{x}^2 > 0$ on Σ_w, and, therefore, the tangent set is empty and the flow is everywhere transverse to Σ_w.

Since the section is defined in velocity space and the kinetic energy never vanishes, the connected components of Σ_w have the same topology as configuration space. The equation $\dot{x}^2 = 2(h - V_0)$ gives two non-zero solutions for $\dot{x}$ for every point in configuration space T^2. Therefore Σ_w consists of two T^2, one with $\dot{x} > 0$ and the other one with $\dot{x} < 0$. Since for the angle $\phi = \arctan(\dot{y}/\dot{x})$ in velocity space we find that $\dot{\phi} = B$ is of constant sign, it is sufficient to consider, for example, the T^2 with $\dot{x} > 0$ as a section. If a W-section yields an ideal section, it necessarily has at least two disjoint components, because the sign of $\dot{S}$, which indicates maxima and minima of the signal, alternates.

If a small periodic potential $V(x, y)$ is introduced, we take $W = \dot{x}/\sqrt{\dot{x}^2 + \dot{y}^2}$ instead of just $W = \dot{x}$ and obtain

$$\dot{W} = S = \dot{y}\frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = 0 \tag{15}$$

as our section condition. We assume $h > V_{\max}$ so that we always have $\dot{x}^2 + \dot{y}^2 > 0$. In order to reduce the section condition to $S = \dot{y} = 0$ alone as before, we furthermore require that

$$\begin{aligned} &\dot{y}\ddot{x} - \dot{x}\ddot{y} = -B(\dot{x}^2 + \dot{y}^2) + (-\dot{x}V_y + \dot{y}V_x) \neq 0 \Leftrightarrow \\ &|B|(\dot{x}^2 + \dot{y}^2) > |-\dot{x}V_y + \dot{y}V_x|. \end{aligned} \tag{16}$$

A sufficient condition for the above inequality to hold is

$$\begin{aligned} &|B|(\dot{x}^2 + \dot{y}^2) > \sqrt{\dot{x}^2 + \dot{y}^2}|\nabla V| \Leftrightarrow \\ &|B|\sqrt{2(h - V)} > |\nabla V| \Leftrightarrow . \end{aligned} \tag{17}$$

This gives the following condition for h:

$$h > \frac{|\nabla V|^2}{2|B|^2} + V. \tag{18}$$

If the last condition holds, we can set $S = \dot{y}$ and the tangent set is given by $\ddot{y} = -B\dot{x} - V_y = 0$ on Σ_w. The tangent set is empty if $h > V_y^2/2B^2 + V$, which is satisfied because already (18) holds. If this condition is violated, the section becomes non-transverse, and eventually for sufficiently small h and/or B also not $\mathcal{E}$-complete: since y is an angle, the section condition $\dot{y} = 0$ will miss the rotating motion (or thinking of the original system as living on $\mathbb{R}^2$, the motion with ever increasing y).

6.2. Birkhoff sections

We obtain examples of Birkhoff sections as a complete W-section with invariant tangent sets. The following example is again due to Birkhoff [2], and we discuss a special case, namely the well known Hénon–Heiles system [4]:

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(x^2 + y^2) + x^2y - \tfrac{1}{3}y^3. \tag{19}$$

If the energy is restricted to the range $(0, \frac{1}{6})$, the dynamics is bounded and $\mathcal{E}_h = S^3$. A zero-force section in x gives

$$S = \dot{p}_x = x(1 + 2y) = 0 \tag{20}$$

where the second factor is always positive because for the above energy range we find $y \in (-\frac{1}{2}, 1)$. Therefore $x = 0$ is a good section condition. The tangent set Ω is given by $\dot{x} = p_x = 0$, describing a stable periodic orbit which moves along the y axis. The tangent set is invariant because $\ddot{S} = \ddot{x} = \dot{p}_x = 0$ is automatically fulfilled on Ω. On the section $x = 0$ the y coordinate varies on an interval. For every interior point of this interval the allowed momenta are given by a circle, while for the endpoints of this interval the circle collapses to a point, i.e. the momenta must be zero. Combining all these circles along the interval we find that $\Sigma_w \simeq S^2$. Cutting this sphere along the invariant tangent set and projecting one half with a constant sign of $\dot{S} = p_x$ onto the canonical plane (y, p_y) we obtain the section of Hénon and Heiles. Note that the zero-velocity section $S = \dot{x} = p_x = 0$ gives a different section which nevertheless has the same invariant tangent set.

This construction works for all bounded potentials with similar properties. Following Birkhoff [2] we assume a Hamiltonian of the above form with a potential that has a straight line (which we choose to be the y axis) on which $\partial V/\partial x = V_x = 0$ holds, and for which V_x and x have equal sign. Then the same construction can be used to obtain a section which is complete and where P is smooth. Another famous example of this type is the hydrogen atom in a magnetic field, see [16] and the reference therein. In this system the orbit in the invariant tangent set can become inverse hyperbolic under parameter variation, but the section nevertheless stays complete and smooth.

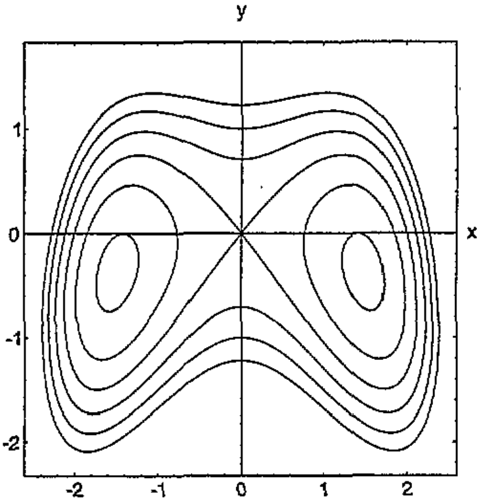


Figure 3. The equipotential lines $V = -0.5, 0, 0.5, 1, 1.5$ for $a = 1$, $\varepsilon = \frac{1}{3}$ in (22). In the integrable case $\varepsilon = 0$ there are two stable periodic orbits parallel to the y axis crossing the minima at $y_0^2 = -\varepsilon x_0^2$, $x_0^2 = a/(1 - 2\varepsilon^2)$. The critical energy of the minima is $a^2/(8\varepsilon^2 - 4)$, for the saddle point at the origin it is 0. The unstable orbit crossing the saddle moves on the y axis for all values of ε. It is also part of the tangent set of the section $\dot{x} = 0$. There are homoclinic orbits associated with this orbit that intersect the section only once and then approach the invariant part of the tangent set for $t \to \pm\infty$.

Yet another well known example, where a Birkhoff section can be established, is the quartic potential

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4}(x^4/b + by^4) + \frac{\gamma}{2}x^2y^2 \tag{21}$$

which has been discussed by e.g. Carnegie and Percival [17] and Eckhardt *et al* [18], who were mainly interested in the limiting behaviour $\gamma \to -1$ ($b = 1$), and Bohigas *et al* [19], who introduced the anisotropy b and discussed a parameter variation $-1 < \gamma < 0$ ($b = \pi/4$). The latter discussion will be reconsidered as the last example. Carnegie, as well as Eckhardt, used $x = 0$ as a section condition, which in our terms is incomplete because there are unstable orbits which never cross the section. Also the tangent set consists of a non-winding unstable periodic orbit. Therefore, there should be serious kinds of discontinuities in the return time as discussed earlier. A simple possibility of avoiding these problems is to consider the transformed system with $x = \tilde{x} - \tilde{y}$ and $y = \tilde{x} + \tilde{y}$ and choose a section condition $\tilde{x} = 0$ as already suggested in [17]. It is easily seen that this section is of Birkhoff type (provided $\gamma < 0$ and $b = 1$).

6.3. Asymptotically complete W-sections

In this and the next example we present the worst cases that can occur for a W-section. The tangent set contains one or more unstable periodic orbits whose invariant manifolds do not intersect Σ_w in the immediate neighbourhood of the orbit. For $\varepsilon = 0$ both systems are integrable.

The first system considered is given by a Hamiltonian $H = p^2/2 + V$ with potential (see figure 3)

$$V = \tfrac{1}{2}(x^4/2 - ax^2 + y^2) + \varepsilon x^2 y \tag{22}$$

$$V_x = x(x^2 - a + 2\varepsilon y). \tag{23}$$

For $\varepsilon^2 < \frac{1}{2}$ the motion is bounded for every h. For $a^2/(2\varepsilon^2 - 1) < 4h < 0$ the accessible region in configuration space Q_h has the topology of two discs; for $h > 0$ we have $Q_h \simeq D^2$ and $\mathcal{E}_h \simeq S^3$. We restrict our attention to the latter case, because then there exists an unstable periodic orbit on the y axis. Consider the zero-velocity section $S = \dot{x} = p_x = 0$. The tangent set is given by $V_x = 0$. Note the change of interpretation of V_x. For the zero-force section $\dot{p}_x = 0$ of the last example it gives the section condition in configuration space, while for a zero-velocity section it gives the tangent set in Σ_w. Nevertheless, the (unstable) periodic orbit on the y axis (for $h > 0$) is part of the tangent set for both sections. Therefore we can have orbits which only asymptotically reach the section, and it is only Σ^∞-complete. Since every homoclinic orbit must have a turning point with $p_x = 0$, the section is $\mathcal{E}$-complete.

If we take the obviously bad section condition $x = 0$, we have an invariant tangent set, but the section is neither $\mathcal{E}$-complete nor Σ-complete. For ε not too large there are stable periodic orbits crossing the minima of the double well around $x = \pm a$. They never intersect $x = 0$ so that the section is not $\mathcal{E}$-complete. We are missing a set of orbits with positive measure. Moreover, there will be a set in Σ_h which approaches part of the set of orbits that do not hit the section. We can picture this situation as a scattering experiment which tries to explore the dynamics around the minima of the double well while sitting on the saddle in between. From chaotic scattering [20, 21] we expect to have a fractal set of singularities in the map. This is indeed the case, as is illustrated in figure 4. Note that also in the integrable case $\varepsilon = 0$ this section is very bad, even though it looks quite nice. The tangent set again is invariant and the map is even smooth on $\Sigma_w \backslash \Lambda$. Cases like this, where the invariant manifolds of an unstable orbit in the tangent set do not meet the section at all, can only occur in integrable systems. The present example was constructed in order to make these possibilities obvious. In this, and in more complicated systems, the use of W-sections can avoid these problems.

The potential in the following example is constructed by the same principle as in the last one, except that it has three minima and two saddles (see figure 5):

$$V(x, y) = x^6 - 2x^4 + x^2 + y^2 + \varepsilon xy(x^2 - 1) \tag{24}$$

$$V_x = (3x^2 - 1)(2x^3 - 2x + \varepsilon y). \tag{25}$$

For $\varepsilon^2 < 4$ the motion is bounded. We restrict our attention to energies above the saddle energy ($h > (4 - \varepsilon^2)/27$), so that $Q_h \simeq D^2$ and $\mathcal{E}_h \simeq S^3$. There exist two unstable periodic orbits crossing the saddle points which are parallel to the y axis at $x = \pm 1/\sqrt{3}$. If we again choose the zero-velocity section $S = \dot{x} = p_x = 0$, both unstable orbits are part of the tangent set. For small ε there are heteroclinic orbits, connecting the two saddles, that only meet $p_x = 0$ in their limit set. Thus this W-section is only $\mathcal{E}^\infty$-complete.

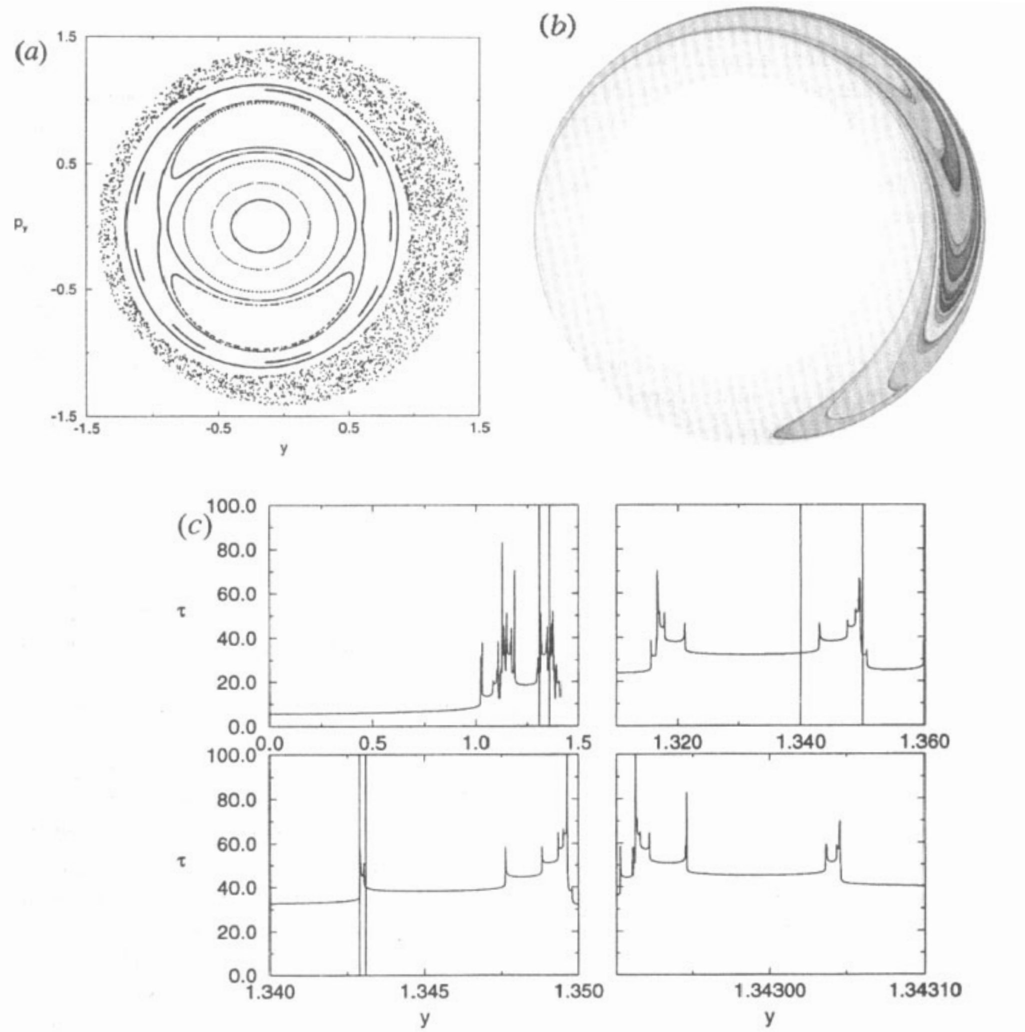


Figure 4. The bad section $(x = 0)$ with $\dot{x} > 0$ in its projection to the (y, p_y)-plane for the Hamiltonian with potential (22) and $\varepsilon = 0.1$, $a = 1$ and $h = 1$. The outer boundary is the invariant tangent set, which consists of an unstable non-winding periodic orbit. Although the Poincaré map in (a) looks quite nice, the plot of the return time $\tau(x)$ in (b) reveals that there is a fractal set of singularities in τ. Dark grey areas correspond to large return time. In part (c) the return time τ is plotted as a function of y with $p_y = 0$ in the section. The enlargements show the fractal nature of the singularity set. For $\varepsilon = 0.1$ there are stable orbits (and therefore a set of orbits with positive measure) in $\mathcal{E}_h$ which do not intersect the section (see figure 6). If there were only unstable orbits missing, the fractal singularity set would still exist, only its scaling properties would be changed.

6.4. Complete W-sections

Note that in the last two examples we chose an unsuitable section condition in order to illustrate what might go wrong. For both potentials a much better choice is $S = \dot{p}_y = -V_y = 0$. In the integrable case $\varepsilon = 0$ this gives a Birkhoff section, because the tangent

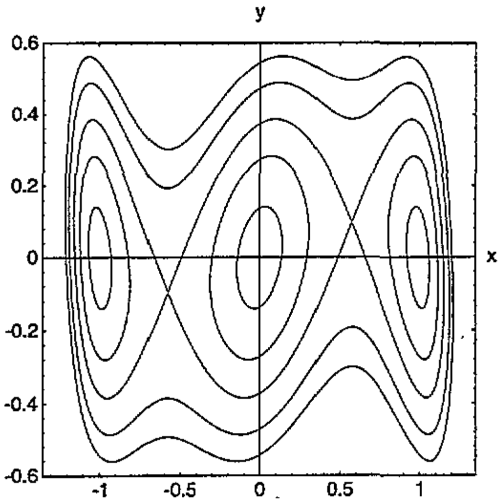


Figure 5. The equipotential lines $27V = 0.5, 2, 3.75, 6, 8$ for $\varepsilon = \frac{1}{2}$ in (24). In the integrable case $\varepsilon = 0$ there are three stable periodic orbits parallel to the y axis crossing the minima at $x = 0, \pm 1$ on the x axis. The potential minima are 0, while the saddle points have $V = (4 - \varepsilon^2)/27$. The perturbation is chosen in such a way that the two unstable orbits crossing the saddles at $x_0 = \pm 1/\sqrt{3}$, $\pm y_0 = \varepsilon x/3$ keep moving parallel to the y axis if $\varepsilon \neq 0$. These orbits are part of the tangent set of the section $\dot{x} = 0$. There are heteroclinic orbits connecting them that do not intersect the section but only approach the invariant part of the tangent set for $t \to \pm\infty$.

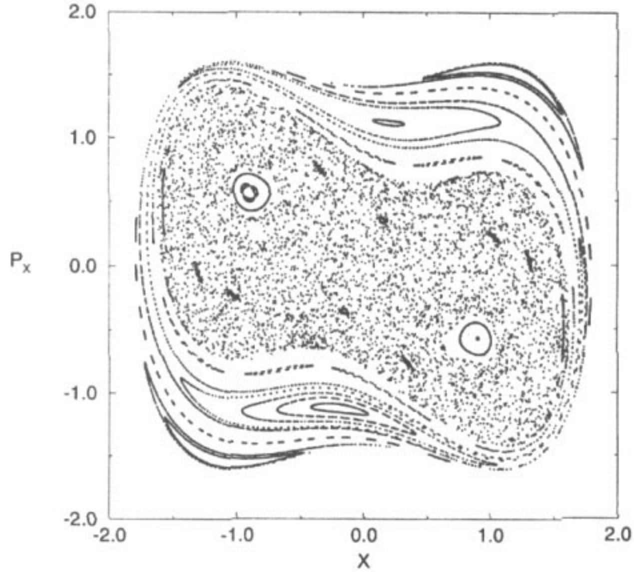
set consists of the elliptic periodic orbit $y = 0$. The section condition can be interpreted as cutting the angle of the action-angle variables for the y motion. Note that this kind of action-angle variable section condition always gives sections with invariant tangent sets. However, as already noted, they cannot be transverse everywhere. Moreover, often action-angle variables cannot be introduced globally by a smooth transformation, and therefore yield a collection of sections. The difference between the section $y = 0$ and $\phi = 0$ is that the latter gives as disk Σ_h (bounded by the invariant set), while the former gives S^2 (divided into two disks by the invariant set). This difference is created by the singularity in the action-angle variables.

In the non-integrable case the section $\dot{p}_y = 0$ is complete. Because the tangent set is not invariant for $\varepsilon > 0$, there are discontinuities in P where τ has a finite jump, as illustrated in figure 6. It is quite typical that a W-section which is a Birkhoff section in the integrable case becomes complete but non-smooth if the perturbation is turned on, because a generic perturbation will destroy the invariance of Ω.

As mentioned above, Birkhoff gave a condition for a potential such that a section in his sense exists. We now present an extension of this construction which yields a complete W-section for potentials with the following property: there exists a straight line, which we again assume to be the y axis, such that $\nabla V \cdot n$ has the same sign as x, where n is a constant vector with a positive x-component. If $n = (1, 0)$ we reobtain Birkhoff's original discussion. We now choose

$$W = p \cdot n \qquad \dot{W} = \dot{p} \cdot n = -\nabla V \cdot n. \tag{26}$$

By assumption $\dot{W}$ will only be zero on the y axis and we can, therefore, choose $S = x = 0$ as a section condition. The tangent set is given by $\dot{x} = p_x = 0$, which in general is not invariant because $\dot{p}_x = -V_x \neq 0$. As an example we want to reconsider the quartic potential



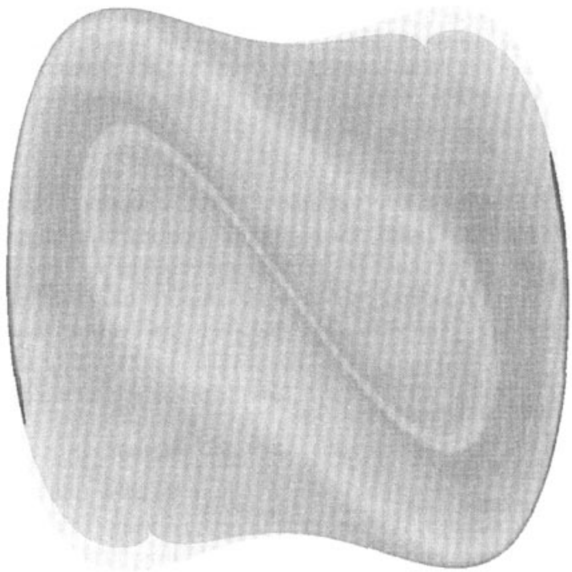


Figure 6. The W-section $\dot{p}_y = 0$ for (22) with $\varepsilon = 0.1$, $a = 1$ and $h = 1$. After a canonical point transformation $Y = V_y = y + \varepsilon x^2$, $X = x$, the projection of Σ_w onto the plane $(X, P_X) = (x, p_x - 2\varepsilon x p_y)$ with $\ddot{p}_y > 0$ gives a canonical representation of P on $\mathbb{R}^2$ which is used here. In (a) some orbits of the Poincaré map P are shown. In (b) the return time τ is illustrated. Dark grey areas correspond to large return times. The maximum return time is ≈ 10, i.e. the section is Σ-complete. Because the tangent set (which corresponds to the boundary of the shown projection of Σ_w) is not invariant, P is discontinuous along the pre-image of Ω, as can be seen in the upper right and in the lower left part of (b), where there is a jump in the grey scale. Although P is discontinuous, invariant tori intersecting the section continue smoothly across the discontinuity as can be seen in (a). The two islands embedded in the chaotic sea correspond to the stable orbits that are lost in the incomplete section shown in figure 4.

(21) in the form of Bohigas *et al* ($b = \pi/4, -1 < \gamma < 0$) [19]. They were aware of the difficulties mentioned earlier and chose to consider the section condition $S = xy = 0$, i.e. $x = 0$ as well as $y = 0$, which together form a complete section. If one wants to use one manifold as a section one can again rotate the coordinate system—this time by an angle of $\tan^2\theta = 1/b$ instead of $\pi/4$ for $b = 1$. In this coordinate system the potential has the properties mentioned above and therefore $S = \tilde{x}$ results in a complete section. Depending on the parameter b, one can do better for a certain range of γ. A rotation of the coordinate system by $\tan^2\theta = (1-\gamma b)/(b(b-\gamma))$ yields a Birkhoff section if $(b^2 + 1 - \sqrt{b^4 + 14b^2 + 1})/2b < \gamma < 0$.

As a last illustrative example for constructing a complete section we choose the double pendulum [22]. It is very simple as a physical system but nevertheless it is more complicated from a topological and dynamical point of view than the systems discussed so far. Without actually producing Poincaré sections on the computer we construct a complete W-section and show how it changes when the topology of the energy surface changes. By way of this example, we show that the use of W-sections is a general procedure which also works for systems with non-trivial topology.

We will adopt the notation from [23]

$$H = \tfrac{1}{2} p^t \mathbf{M}^{-1} p + V(\phi_1, \phi_2) \tag{27}$$

$$V = \gamma_1(1 - \cos\phi_1) + \gamma_2(1 - \cos\phi_2) \tag{28}$$

$$\mathbf{M} = \begin{pmatrix} \alpha & \varepsilon\cos(\phi_2 - \phi_1) \\ \varepsilon\cos(\phi_2 - \phi_1) & \beta \end{pmatrix} \tag{29}$$

where $\alpha\beta > \varepsilon^2$ and we assume $\gamma_1 > \gamma_2$. The configuration space is T^2. The potential is shown in figure 7.

There are four critical energies corresponding to the four equilibrium points $(\phi_1, \phi_2) = (0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π). We denote the corresponding critical energies by $h_0 = 0$, $h_1 = 2\gamma_2$, $h_2 = 2\gamma_1$, and $h_3 = 2(\gamma_1 + \gamma_2)$. The accessible region of configuration space Q_h is determined by setting $V = h$. We denote it by Q_i for $h_{i-1} < h < h_i$ and find, by inspecting figure 7, $Q_0 = \emptyset$, $Q_1 \simeq D^2$, $Q_2 \simeq S^1 \times D^1$, $Q_3 \simeq T^2 \backslash D^2$ (a torus with a disk removed) and $Q_4 \simeq T^2$ (with $h_{-1} = -\infty$, $h_4 = \infty$). Using Smale's construction for the energy surface [14] we find $\mathcal{E}_{h_1} = S^3$, $\mathcal{E}_{h_2} = S^2 \times S^1$, $\mathcal{E}_{h_3} = K^3$ $(= S^1 \times S^2 \# S^1 \times S^2)$, and $\mathcal{E}_{h_4} = T^3$.

Since ϕ_i can rotate, $\dot{\phi}_i = 0$ is not a W-section, at least not for $h > h_2$. If we instead take $W = 1 - \cos\phi_1$, which amounts to measuring the y coordinate of a point on the inner pendulum, this signal will obviously have an infinite number of extrema for almost every trajectory. The section condition now becomes

$$S = \dot{W} = \dot{\phi}_1 \sin\phi_1 = \frac{(\beta p_1 - \varepsilon\cos(\phi_2 - \phi_1) p_2)}{\det\mathbf{M}} \sin\phi_1 = 0. \tag{30}$$

Σ_w is not a smooth manifold in this case. It consists of three manifolds Σ_l, Σ_0 and Σ_π, corresponding to $S_l = \dot{\phi}_1 = 0$, $S_0 = \phi_1 = 0$, and $S_\pi = \phi_1 = \pi$, of which Σ_l and Σ_0, and Σ_l and Σ_π, respectively, intersect in phase space.

For $h < h_2$ the inner pendulum cannot turn over. Therefore ϕ_1 maps $\mathcal{E}_h$ to an interval and S_l by itself is a W-section. For higher energies we additionally have to consider S_0 (or S_π), because otherwise we would miss the rotating motion. It is not necessary to consider both S_0 and S_π, because if ϕ_1 is rotating it is caught by either one, and if it is not rotating it is caught by S_l.

Depending on the energy range $h_{i-1} < h < h_i$ we denote the corresponding section by Σ_{li} and $\Sigma_{\pi i}$, respectively. Σ_{li} is a double covering of Q_i because for every point in Q_i

there are two values for $\dot{\phi}_2$. For $h < h_2$ the inner pendulum cannot reach π and therefore $\Sigma_{\pi 1} = \Sigma_{\pi 2} = \emptyset$. In the third range ϕ_2 can vary on an interval if $\phi_1 = \pi$ is fixed and for $h > h_3$ we of course find $\phi_2 \in S^1$. Collecting these results gives the following table (where R_2^2 denotes a Riemann surface of genus 2):

i	0	1	2	3	4
h	<0	$(0, h_1)$	(h_1, h_2)	(h_2, h_3)	$> h_3$
Q_i		D^2	$S^1 \times D^1$	$T^2 \backslash D^2$	T^2
$\mathcal{E}_{h_i}$		S^3	$S^2 \times S^1$	K^3	T^3
Σ_{li}		S^2	T^2	R_2^2	$2T^2$
$\Sigma_{\pi i}$				S^2	T^2

(31)

We choose to project Σ_π onto the cylinder (ϕ_2, p_2), which we picture on a plane with periodic boundary conditions in ϕ_2. We obtain a double cover of a region in this plane, because the equation $H = h$ is quadratic in p_1. At the boundary of this region the two solutions for p_1 collapse onto each other. The mapping in (ϕ_2, p_2) is area preserving. Calculating the boundary of the projection of Σ_π onto the plane gives

$$p_2^2 = 2(h - V(\pi, \phi_2)) \det \mathbf{M}/\alpha \tag{32}$$

which coincides with the tangent set Ω, which is not invariant under the flow ($\Lambda = \emptyset$). Pictures of Σ_0, which are constructed by the same procedure can be found in [22].

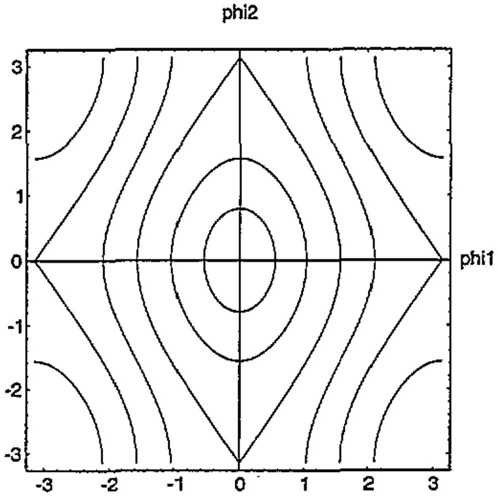


Figure 7. The equipotential lines $V = 0.5, 1, 2, 3, 4, 5$ for the mathematical double pendulum, i.e. $\alpha = 2$, $\beta = 1$, $\gamma_1 = 2$, $\gamma_2 = 1$ in (28). The critical energies correspond to the origin ($h = 0$), the cusps on the ϕ_2 axis ($h = 2$), the cusps on the ϕ_1 axis ($h = 4$) and the four corners of the picture ($h = 6$), which are identified to one point on the torus. Because the kinetic energy depends non-trivially on q, there are no periodic orbits along the symmetry lines of the potential.

The section condition $S_l = 0$ is linear in the momenta. Projecting Σ_l onto configuration space gives Q_h. For every interior point of Q_h there are two possible momenta while for a point in ∂Q_h the two solutions collapse. The boundary of this projection as given by $V(\phi_1, \phi_2) = h$ does not coincide with Ω. The tangent set of Σ_l can be described by a fourth-order polynomial in $\cos\phi_2$ and p_2^2. Except for the integrable case without potential (or equivalently $h \to \infty$), it turns out not to be invariant under the flow.

As a result of the foregoing discussion we conclude that the Poincaré section $\dot{W} = \dot{\phi}_1 \sin\phi_1$ for the double pendulum is complete, but not transverse and for $h > h_2$ it is not a smooth manifold. Similarly, one can construct a W-section by considering $W = \cos(\phi_2 - \phi_1)$. $\dot{W} = -(\dot{\phi}_2 - \dot{\phi}_1)\sin(\phi_2 - \phi_1)$ generates a complete section. A section condition like $\phi_2 = \phi_1$ [24] is not complete, because there exist unstable periodic orbits for which the double pendulum rotates close to the folded configuration, i.e. with $\phi_2 - \phi_1 \approx \pi$. Most notably this section does not even contain some of the equilibrium points, which should always be taken as an indication of the incompleteness of the section. The reduction of one pendulum to action angle variables in the limit $\varepsilon \ll \alpha\beta$ as described in [25] shows that in this limit the section $\phi_1 = 0$ is complete. But this argument based on perturbation theory is valid only far away from the 'mathematical double pendulum' [24] with $\alpha = 2$, $\beta = 1$, $\gamma_1 = 2$, $\gamma_2 = 1$, and $\varepsilon = 1$.

7. Generalizations

We have restricted our attention to Hamiltonian systems with two degrees of freedom, since on the one hand $\mathcal{E}_h$ is topologically more interesting than $\mathbb{R}^3$ (for dissipative flows) and on the other hand they are simple enough because the invariant sets in Λ are just periodic orbits. In principle the discussion also applies to higher-dimensional Hamiltonian systems and to non-Hamiltonian flows on orientable manifolds.

7.1. Higher dimensions

The basic idea of the construction of complete W-sections works for any number of degrees of freedoms of the Hamiltonian as the proof of the theorem, which makes no assumptions about the dimensionality, shows. Take the vanishing of the derivative of a compact function as a section condition and show that the assumptions of corollaries 1 or 2 are fulfilled.

Corollary 3 cannot be transferred as easily. First of all, one must generalize the notion of winding. A more serious problem is that invariant sets that live on higher than two-dimensional surfaces Σ_h can be very complicated, while for the case studied above they can only be composed of periodic orbits. Birkhoff sections are difficult/impossible to obtain, because one needs invariant subspaces of codimension three in phase space. The most promising approach to constructing a W-section, therefore, is to choose a function W such as to avoid invariant tangent sets. This should easily be possible and the section then is necessarily complete.

If the Hamiltonian system is described by a Poisson structure with k Casimir constants C_i such that the phase space $\mathcal{P}$ has dimension $k+4$, the above analysis has to be modified a little, in order to take care of the Casimir constants. For example, the section is now given by

$$\Sigma_c = \{x \in \mathcal{P} | H = h, S = 0, C_i = c_i, i = 1, \ldots, k\}. \tag{33}$$

In order to find out whether Σ_c is a manifold the rank of the $(k+2) \times (k+4)$ Jacobian has to be considered. In this way the algebra can become much more challenging, but the basic ideas can all be applied. For an application of this to the Kovalevskaya top see [26, 25].

7.2. Non-Hamiltonian systems

The application of W-sections to dissipative flows (e.g. in $\mathbb{R}^3$) is straightforward. Assume there exists a finite trapping region, i.e. a volume U on whose boundary the vector field is transverse and pointing inward. A W-section is then considered inside this volume. Any function of the phase space variables, especially just one of the variables itself, now is a candidate for W. For dissipative flows on $\mathbb{R}^3$, Σ-completeness for the inverse map cannot be achieved by our construction, but for these systems the time reversed dynamics is usually not considered. The discussion of the effect of periodic orbits in Λ has to be extended, because in section 4 we only covered the Hamiltonian case.

As stated in the beginning, it is impossible to get an ideal section. Besides the given topological argument, in the dissipative case this is most obvious if the flow has at least one equilibrium point x^* in D^3. If $x^* \not\in \Sigma$ then it is not $\mathcal{E}$-complete; if $x^* \in \Sigma$ then it is not transverse because of $\xi(x^*) = 0$. Note that by construction a W-section contains all the equilibrium points. Choosing coordinates on Σ is usually much simpler for a flow in $U \subset \mathbb{R}^3$, since we typically do not get closed surfaces, if we intersect Σ with U, but instead often just D^2.

An almost trivial example is given by differential equations with one linear component: $\dot{x} = F(x, y, z)$, $\dot{y} = G(x, y, z)$, $\dot{z} = ax + by + cz + d$. Assume that the flow is pointing inward on a sphere, which is the surface of a sufficiently large ball. The section condition now is $S = \dot{z} = ax + by + cz + d = 0$, which is a plane in phase space and the intersection with the ball inside the sphere gives a disk D^2. Since one of a, b, c is non-zero, (assume $c \neq 0$) we take x and y as coordinates on Σ. Similarly if $\dot{z}$ is at most quadratic the section becomes a quadric in $\mathbb{R}^3$.

8. Conclusions

Our aim was to construct complete sections for Hamiltonian systems with two degrees of freedom. The method of W-sections can be of great practical use for this purpose, as we have shown in the examples.

First, we have given some negative results by showing that there can be topological obstacles. If the energy surface is S^3 there does not exist a globally transverse section. Moreover, we have shown that for systems with time reversal symmetry there can never exist a time symmetric transverse section with only one component. Thus this leads to the discussion of non-transverse sections.

Non-transverse sections can suffer from three types of discontinuities, related to the properties of the tangent set Ω. If the tangent set is invariant under the flow and the periodic orbits in this set are winding, one is lead to the type of sections that were introduced by Birkhoff. If the tangent set contains non-winding orbits and/or non-invariant components the Poincaré map becomes discontinuous. If the section is not even $\mathcal{E}^\infty$-complete, there can be a fractal set of singularities in P. That the discussion of these discontinuities has not been a topic in the literature so far seems to be due to two reasons. The section conditions are often obtained by symmetry considerations, which naturally yield an invariant tangent set being an elliptic periodic orbit, and therefore Birkhoff sections, which is the best one can hope for in general; the second reason is that looking at a Poincaré section does not necessarily show the difficulties arising from discontinuities, which are restricted to a set of measure zero. Only the investigation of the return time clearly shows the smoothness properties of the Poincaré map, as was illustrated in the examples.

Of course, one should always try to obtain smooth Poincaré maps, i.e. ideal or Birkhoff section. Our new method of W-sections can be of help in finding these favourable sections. We have demonstrated this by generating an ideal section for a particle in a periodic magnetic field with a small periodic potential. For Birkhoff sections we have discussed well known systems and found a smooth section for the quartic potential that has not been used in the literature.

If there is no symmetry in the problem, and therefore no explicitly given periodic orbit, it can be practically impossible to obtain a Birkhoff section, even though it might exist in principle. Taking these difficulties into account, we end up restricting attention to the main property demanded of a section in practical applications, which is (asymptotical) completeness, where (asymptotical) $\mathcal{E}$-completeness is more important, because it implies Σ^∞-completeness.

We have shown that our method of W-section automatically generates sections that are asymptotically complete. The main idea was to consider a signal $W(\Phi^t(x))$ which must be a bounded quantity, and to take its time derivative (i.e. the extrema of $W(\Phi^t(x))$) as a section condition. The resulting section is asymptotically complete, because the limit sets of orbits with infinite return time are in the invariant tangent set. Depending on the properties of the tangent set we distinguish five types of W-sections.

There are a number of standard choices for W-sections from which the zero-force, velocity, and acceleration type are illustrated in the examples. If there are non-winding orbits in the invariant tangent set, the section is only asymptotically complete. This rather unpleasant possibility is also illustrated in the examples, but our ultimate goal is the construction of complete W-sections. Then the induced Poincaré map never has infinite return time τ. There is a discontinuity arising from the non-invariant part of the tangent set. This type of discontinuity with a finite jump in τ does not disturb the appearance of the section, because the continuity of the flow in phase space ensures that (e.g. KAM) tori crossing this discontinuity nevertheless are smooth. This type of complete W-section has been applied to the double well, the double pendulum, and a modified quartic potential.

Acknowledgments

We thank P H Richter, A Bolsinov, and J Wiersig for inspiring discussions. Also we are grateful for some helpful suggestions by the referees. The authors acknowledge support by the Deutsche Forschungsgemeinschaft.

References

[1] Poincaré H 1892 *Les Méthodes Nouvelles de la Mécanique Céleste* (Paris: Gauthier-Villars) (translated by the AIP 1993 *History of Modern Physics and Astronomy* vol 13)
[2] Birkhoff G D 1917 *Trans. Am. Math. Soc.* **18** 199
[3] Birkhoff G D 1922 *Acta Math* **43** 1
[4] Hénon M and Heiles C 1964 *Astron. J.* **69** 73
[5] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Berlin: Springer)
[6] Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Dynamics* (Berlin: Springer)
[7] Ozorio de Almeida A M 1988 *Hamiltonian Systems: Chaos and Quantization* (Cambridge: Cambridge University Press)
[8] Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd edn (Reading, MA: Benjamin-Cummings)
[9] Palis J and de Melo W 1982 *Geometric Theory of Dynamical Systems* (Berlin: Springer)
[10] Hénon M 1982 *Physica* **5D** 412
[11] Siegel C L and Moser J K 1972 *Lectures on Celestial Mechanics* (Berlin: Springer)
[12] Birkhoff G D 1922 *Dynamical Systems* (Providence, RI: American Mathematical Society)
[13] Nitecki Z 1971 *Differentiable Dynamics* (Cambridge, MA: MIT Press)
[14] Smale S 1970 *Inv. Math.* **10** 305
[15] Lorenz E 1963 *J. Atmos. Sci.* **20** 130
[16] Friedrich H and Wintgen D 1989 *Phys. Rep.* **183** 37
[17] Carnegie A and Percival I C 1984 *J. Phys. A: Math. Gen.* **17** 801

[18] Eckhardt B, Llorente J M G and Pollak E 1990 *Chem. Phys. Lett.* **174** 325

[19] Bohigas O, Tomsovic S and Ullmo D 1993 *Phys. Rep.* **223** 43

[20] Jung C and Eckhardt B 1986 *J. Phys. A: Math. Gen.* **19** L829

[21] Jung C and Scholz H-J 1988 *J. Phys. A: Math. Gen.* **21** 2301

[22] Richter P H and Scholz H-J *Stochastic Phenomena and Chaotic Behaviour in Complex Systems* ed P Schuster (Berlin: Springer) pp 86–97

[23] Dullin H R 1994 *Z. Phys.* B **93** 521.

[24] Richter P H and Scholz H-J 1986 Publikationen zu Wissenschaftlichen Filmen, Sektion Technische Wissenschaften/Naturwissenschaften Serie 9 3, Film C1574, Institut für den Wissenschaftlichen Film (IWF), Göttingen

[25] Dullin H R 1994 *Die Energieflächen des Kowalewskaja-Kreisels* (Aachen: Mainz) Dissertation

[26] Dullin H R, Juhnke M and Richter P H 1994 *Bifurcation and Chaos* **4** 1535